PERIODIC OSCILLATIONS OF QUASILINEAR AUTONOMOUS SYSTEMS WITH TWO DEGREES OF FREEDOM

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1. We shall study a quasilinear vibration system with two degrees of freedom

$$\begin{aligned} a_{11}\ddot{x}_1 + a_{12}\ddot{x}_2 + c_{11}x_1 + c_{12}x_2 &= \mu F_1(x_1, x_2, \dot{x}_1, \dot{x}_2, \mu) \\ a_{21}\ddot{x}_1 + a_{22}\ddot{x}_2 + c_{21}x_1 + c_{22}x_2 &= \mu F_2(x_1, x_2, \dot{x}_1, \dot{x}_2, \mu) \end{aligned} \qquad (1.1)$$

The functions F_1 and F_2 are assumed to be analytic in their arguments in some region of their variation. The quantity μ is a small parameter. The generating system (with $\mu = 0$) is a linear conservative system with constant coefficients, where $a_{12} = a_{21}$, $c_{12} = c_{21}$.

Let us assume that the frequency equation of the generating system

$$\begin{vmatrix} c_{11} - \omega^2 a_{11} & c_{12} - \omega^2 a_{12} \\ c_{21} - \omega^2 a_{21} & c_{22} - \omega^2 a_{22} \end{vmatrix} = 0 \qquad (1.2)$$

has only positive roots. There are three cases possible: the vibration frequencies are different and commensurate, different and non-commensurate, and equal.

2. Let us look more closely at the case of different and commensurate frequencies. Let $m_1\omega_1 = m_2\omega_2$ where m_1 and m_2 are positive integers. In this case there exists a periodic solution of the generating system with a frequency ω_0 and a period T_0

$$\omega_0 = \frac{\omega_1}{m_2} = \frac{\omega_2}{m_1}, \qquad T_0 = \frac{2\pi}{\omega_0}$$

Let us assume that the original nonlinear system (1.1) has a periodic solution with a period $T = T_0 + \alpha$, which becomes the generating solution with $\mu = 0$. Let us construct this solution.

The solution of the generating system can be represented in the form

$$x_{10}(t) = A_0 \cos \omega_1 t + \frac{B_0}{\omega_1} \sin \omega_1 t + E_0 \cos \omega_2 t$$
$$x_{20}(t) = p_1 \left(A_0 \cos \omega_1 t + \frac{B_0}{\omega_1} \sin \omega_1 t \right) + p_2 E_0 \cos \omega_2 t \qquad (2.1)$$

Terms with $\sin \omega_2 t$ do not enter the solution for an appropriate choice of measuring time t. The values p_1 and p_2 are determined by the formulas

$$p_r = -\frac{c_{11} - \omega_r^2 a_{11}}{c_{12} - \omega_r^2 a_{12}} = -\frac{c_{21} - \omega_r^2 a_{21}}{c_{22} - \omega_r^2 a_{22}} \qquad (r = 1,\ 2) \qquad (2.2)$$

As shown in [3], the initial conditions for the system (1.1) will be

$$x_1(0) = A_0 + \beta_1 + E_0 + \beta_3, \qquad \dot{x}_1(0) = B_0 + \beta_2$$
$$x_2(0) = p_1(A_0 + \beta_1) + p_2(E_0 + \beta_3), \qquad \dot{x}_2(0) = p_1(B_0 + \beta_2) \qquad (2.3)$$

The quantities β_1, β_2 and β_3 are functions of μ which vanish at $\mu = 0$. Then, according to [3] the solution of the original system (1.1) can be represented in the form

$$x_1(t) = x^{(1)}(t) + x^{(2)}(t), \qquad x_2(t) = p_1 x^{(1)}(t) + p_2 x^{(2)}(t) \qquad (2.4)$$

The expansions of $x^{(1)}(t)$ and $x^{(2)}(t)$ in powers of the parameters β_1, β_2, β_3 and μ are of the form

$$x^{(1)}(t) = (A_0 + \beta_1) \cos \omega_1 t + \frac{B_0 + \beta_2}{\omega_1} \sin \omega_1 t + \qquad (2.5)$$
$$+ \sum_{n=1}^{\infty} \left[C_n^{(1)}(t) + \frac{\partial C_n^{(1)}}{\partial A_0} \beta_1 + \frac{\partial C_n^{(1)}}{\partial B_0} \beta_2 + \frac{\partial C_n^{(1)}}{\partial E_0} \beta_3 + \frac{1}{2} \frac{\partial^2 C_n^{(1)}}{\partial A_0^2} \beta_1^2 + \ldots \right] \mu^n$$
$$x^{(2)}(t) = (E_0 + \beta_3) \cos \omega_2 t +$$
$$+ \sum_{n=1}^{\infty} \left[C_n^{(2)}(t) + \frac{\partial C_n^{(2)}}{\partial A_0} \beta_1 + \frac{\partial C_n^{(2)}}{\partial B_0} {}_2 + \frac{\partial C_n^{(2)}}{\partial E_0} \beta_3 + \frac{1}{2} \frac{\partial^2 C_n^{(2)}}{\partial A_0^2} \beta_1^2 + \ldots \right] \mu^n$$

The values of $C_n^{(1)}(t)$ and $C_n^{(2)}(t)$ are determined by means of the formulas

$$C_n^{(1)}(t) = \frac{1}{\Delta_0 (\omega_2^2 - \omega_1^2) \omega_1} \int_0^t R_n^{(1)}(t') \sin \omega_1 (t - t')\, dt'$$
$$C_n^{(2)}(t) = \frac{1}{\Delta_0 (\omega_1^2 - \omega_2^2) \omega_2} \int_0^t R_n^{(2)}(t') \sin \omega_2 (t - t')\, dt' \qquad (2.6)$$

where

$$R_n^{(r)}(t) = (c_{22} - \omega_r^2 a_{22}) H_{1n}(t) - (c_{12} - \omega_r^2 a_{12}) H_{2n}(t) \qquad (r = 1,2) \qquad (2.7)$$
$$\Delta_0 = a_{11} a_{22} - a_{12} a_{21} \qquad (2.8)$$

The following functions will also be used below:

$$C_{1n}(t) = C_n^{(1)}(t) + C_n^{(2)}(t), \qquad C_{2n}(t) = p_1 C_n^{(1)}(t) + p_2 C_n^{(2)}(t) \tag{2.9}$$

The values $H_{1n}(t)$ and $H_{2n}(t)$ which enter Formula (2.7) are equal to

$$H_{in}(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}F}{d\mu^{n-1}}\right)_{\mu=0} \qquad (i=1,2) \tag{2.10}$$

Explicitly, the first three functions of $H_{in}(t)$ are

$$H_{i1}(t) = F_i(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}, 0)$$

$$H_{i2}(t) = \left(\frac{\partial F_i}{\partial x_1}\right)_0 C_{11} + \left(\frac{\partial F_i}{\partial \dot{x}_1}\right)_0 \dot{C}_{11} + \left(\frac{\partial F_i}{\partial x_2}\right)_0 C_{21} + \left(\frac{\partial F_i}{\partial \dot{x}_2}\right)_0 \dot{C}_{21} + \left(\frac{\partial F_i}{\partial \mu}\right)_0$$

$$H_{i3}(t) = \frac{1}{2}\left(\frac{\partial^2 F_i}{\partial x_1^2}\right)_0 C_{11}^2 + \frac{1}{2}\left(\frac{\partial^2 F_i}{\partial \dot{x}_1^2}\right)_0 \dot{C}_{11}^2 + \frac{1}{2}\left(\frac{\partial^2 F_i}{\partial x_2^2}\right)_0 C_{21}^2 +$$
$$+ \frac{1}{2}\left(\frac{\partial^2 F_i}{\partial \dot{x}_2^2}\right)_0 \dot{C}^2_{21} + \frac{1}{2}\left(\frac{\partial^2 F_i}{\partial \mu^2}\right)_0 + \left(\frac{\partial^2 F_i}{\partial x_1 \partial \dot{x}_1}\right)_0 C_{11}\dot{C}_{11} + \left(\frac{\partial^2 F_i}{\partial x_2 \partial \dot{x}_2}\right)_0 C_{21}\dot{C}_{21} +$$
$$+ \left(\frac{\partial^2 F_i}{\partial x_1 \partial x_2}\right)_0 C_{11} C_{21} + \left(\frac{\partial^2 F_i}{\partial x_1 \partial \dot{x}_2}\right)_0 C_{11}\dot{C}_{21} + \left(\frac{\partial^2 F_i}{\partial \dot{x}_1 \partial x_2}\right)_0 \dot{C}_{11} C_{21} + \left(\frac{\partial^2 F_i}{\partial \dot{x}_1 \partial \dot{x}_2}\right)_0 \dot{C}_{11}\dot{C}_{21} +$$
$$+ \left(\frac{\partial^2 F_i}{\partial x_1 \partial \mu}\right)_0 C_{11} + \left(\frac{\partial^2 F_i}{\partial \dot{x}_1 \partial \mu}\right)_0 \dot{C}_{11} + \left(\frac{\partial^2 F_i}{\partial x_2 \partial \mu}\right)_0 C_{21} + \left(\frac{\partial^2 F_i}{\partial \dot{x}_2 \partial \mu}\right)_0 \dot{C}_{21} +$$
$$+ \left(\frac{\partial F_i}{\partial x_1}\right)_0 C_{12} + \left(\frac{\partial F_i}{\partial \dot{x}_1}\right)_0 \dot{C}_{12} + \left(\frac{\partial F_i}{\partial x_2}\right)_0 C_{22} + \left(\frac{\partial F_i}{\partial \dot{x}_2}\right)_0 \dot{C}_{22} \tag{2.11}$$

3. The conditions of periodicity for $x_1(t)$, $x_2(t)$ and their first derivatives will be

$$x_1(T_0+\alpha) = A_0 + \beta_1 + E_0 + \beta_3, \qquad \dot{x}_1(0) = B_0 + \beta_2$$
$$x_2(T_0+\alpha) = p_1(A_0+\beta_1) + p_2(E_0+\beta_3), \qquad \dot{x}_2(0) = p_1(B_0+\beta_2) \tag{3.1}$$

One of these conditions, for instance the periodicity condition for $x_1(t)$, will be used for the determination of the parameter α as an implicit function of the remaining parameters

$$\alpha = \alpha(\beta_1, \beta_2, \beta_3, \mu)$$

We shall seek the value of α in the form of a series in integral powers of these functions. Since α approaches zero as $\mu = 0$, and since the derivatives of any order of α with respect to β_1, β_2 and β_3 are equal to zero for $t = T_0$ and $\mu = 0$, the expansion of α has the form

$$\alpha = \sum_{n=1}^{\infty}\left[N_n(T_0) + \frac{\partial N_n}{\partial A_0}\beta_1 + \frac{\partial N_n}{\partial B_0}\beta_2 + \frac{\partial N_n}{\partial E_0}\beta_3 + \frac{1}{2}\frac{\partial^2 N_n}{\partial A_0^2}\beta_1^2 + \ldots\right]\mu^n \tag{3.2}$$

By successively differentiating the equation $\dot{x}_1(T_0 + \alpha) = B_0 + \beta_2$ with respect to μ we find

$$\left(\frac{\partial\alpha}{\partial\mu}\right)_0 = \frac{1}{P_1}\dot{C}_{11}(T_0) = N_1(T_0) \tag{3.3}$$

$$\left(\frac{\partial^2\alpha}{\partial\mu^2}\right)_0 = \frac{2}{P_1}\left[\dot{C}_{12}(T_0) + \ddot{C}_{11}(T_0)N_1(T_0) - \frac{1}{2}B_0\omega_1^2N_1^2(T_0)\right] = 2N_2(T_0)$$

$$\left(\frac{\partial^3\alpha}{\partial\mu^3}\right)_0 = \frac{6}{P_1}\left[\dot{C}_{13}(T_0) + \ddot{C}_{12}(T_0)N_1(T_0) + \frac{1}{2}\dddot{C}_{11}(T_0)N_1^2(T_0) + \frac{1}{6}Q_1N_1^3(T_0) - \right.$$
$$\left. - B_0\omega_1^2N_1(T_0)N_2(T_0) + \ddot{C}_{11}(T_0)N_2(T_0)\right] = 6N_3(T_0) \quad \text{etc.}$$

where

$$P_1 = A_0\omega_1^2 + E_0\omega_2^2, \qquad Q_1 = A_0\omega_1^4 + E_0\omega_2^4$$

The condition for the existence of the expansion (3.2) is the inequality $P_1 \neq 0$.

By expanding in terms of the parameter α the left-hand sides of the remaining periodicity conditions and substituting into them the α's from Formula (3.2) we obtain for j = 1, 2, 3

$$\sum_{n=1}^{\infty}\left[M_{jn}(T_0) + \frac{\partial M_{jn}}{\partial A_0}\beta_1 + \frac{\partial M_{jn}}{\partial B_0}\beta_2 + \frac{\partial M_{jn}}{\partial E_0}\beta_3 + \frac{1}{2}\frac{\partial^2 M_{jn}}{\partial A_0^2}\beta_1^2 + \ldots\right]\mu^n = 0 \tag{3.4}$$

Now we compute in all three conditions the first three coefficients of the powers of the parameter μ. The coefficients of the first power of μ are

$$\begin{aligned} M_{11}(T_0) &= C_{11}(T_0) + B_0N_1(T_0) \\ M_{21}(T_0) &= C_{21}(T_0) + p_1B_0N_1(T_0) \\ M_{31}(T_0) &= \dot{C}_{21}(T_0) - P_2N_1(T_0) \end{aligned} \tag{3.5}$$

The coefficients of the second power of μ are

$$\begin{aligned} M_{12}(T_0) &= C_{12}(T_0) + B_0N_2(T_0) + \frac{1}{2}P_1N_1^2(T_0) \\ M_{22}(T_0) &= C_{22}(T_0) + p_1B_0N_2(T_0) + \frac{1}{2}P_2N_1^2(T_0) \\ M_{32}(T_0) &= \dot{C}_{22}(T_0) - P_2N_2(T_0) + \ddot{C}_{21}(T_0)N_1(T_0) - \frac{1}{2}p_1B_0\omega_1^2N_1^2(T_0) \end{aligned} \tag{3.6}$$

The coefficients of the third power of μ are

$$M_{13}(T_0) = C_{13}(T_0) + B_0N_3(T_0) + P_1N_1(T_0)N_2(T_0) - \frac{1}{2}\ddot{C}_{11}(T_0)N_1^2(T_0) + \frac{1}{6}B_0\omega_1^2N_1^3(T_0) \tag{3.7}$$

$$M_{23}(T_0) = C_{23}(T_0) + p_1B_0N_3(T_0) + P_2N_1(T_0)N_2(T_0) - \frac{1}{2}\ddot{C}_{21}(T_0)N_1^2(T_0) +$$

$$+\frac{1}{3}p_1B_0\omega_1^2N_1^3(T_0)$$

$$M_{33}(T_0)=\dot{C}_{23}(T_0)-P_2N_3(T_0)-p_1B_0\,\omega_1^2N_1(T_0)\,N_2(T_0)+\ddot{C}_{21}(T_0)\,N_2(T_0)+$$
$$+\ddot{C}_{22}(T_0)\,N_1(T_0)+\frac{1}{2}\ddot{C}_{21}(T_0)\,N_1^2(T_0)+\frac{1}{6}Q_2N_1^3(T_0)$$

The following notation was used in the above formulas:

$$P_2=p_1A_0\omega_1^2+p_2E_0\,\omega_2^2,\qquad Q_2=p_1A_0\omega_1^4+p_2E_0\omega_2^4$$

4. Let us assume that the parameters β_1, β_2 and β_3 can be expanded in series

$$\beta_1=\sum_{n=1}^{\infty}A_n\mu^n,\qquad \beta_2=\sum_{n=1}^{\infty}B_n\mu^n,\qquad \beta_3=\sum_{n=1}^{\infty}E_n\mu^n \tag{4.1}$$

Substitute the values of these parameters into Formula (3.4) and equate to zero the coefficients of equal powers of μ. Equating to zero the coefficients of the first power of μ yields in all three conditions

$$M_{11}(T_0)=0,\qquad M_{21}(T_0)=0,\qquad M_{31}(T_0)=0 \tag{4.2}$$

From these three equations the coefficients A_0, B_0 and E_0 are obtained. We shall call these equations the equations of the fundamental amplitudes.

By equating to zero the coefficients of the second powers of μ we obtain

$$\begin{aligned}
&M_{12}(T_0)+A_1\frac{\partial M_{11}}{\partial A_0}+B_1\frac{\partial M_{11}}{\partial B_0}+E_1\frac{\partial M_{11}}{\partial E_0}=0\\
&M_{22}(T_0)+A_1\frac{\partial M_{21}}{\partial A_0}+B_1\frac{\partial M_{21}}{\partial B_0}+E_1\frac{\partial M_{21}}{\partial E_0}=0\\
&M_{32}(T_0)+A_1\frac{\partial M_{31}}{\partial A_0}+B_1\frac{\partial M_{31}}{\partial B_0}+E_1\frac{\partial M_{31}}{\partial E_0}=0
\end{aligned} \tag{4.3}$$

If the Jacobian

$$\frac{D(M_{11},\,M_{21},\,M_{31})}{D(A_0,\,B_0,\,E_0)}\neq 0 \tag{4.4}$$

then one can determine the coefficients A_1, B_1 and E_1 from Equations (4.3).

By equating to zero the coefficients of the third powers of μ we obtain

$$M_{13}+A_2\frac{\partial M_{11}}{\partial A_0}+B_2\frac{\partial M_{11}}{\partial B_0}+E_2\frac{\partial M_{11}}{\partial E_0}+\frac{1}{2}A_1^2\frac{\partial^2 M_{11}}{\partial A_0^2}+\frac{1}{2}B_1^2\frac{\partial^2 M_{11}}{\partial B_0^2}+\frac{1}{2}E_1^2\frac{\partial^2 M_{11}}{\partial E_0^2}+$$
$$+A_1B_1\frac{\partial^2 M_{11}}{\partial A_0\,\partial B_0}+A_1E_1\frac{\partial^2 M_{11}}{\partial A_0\,\partial E_0}+B_1E_1\frac{\partial^2 M_{11}}{\partial B_0\,\partial E_0}+A_1\frac{\partial M_{12}}{\partial A_0}+B_1\frac{\partial M_{12}}{\partial B_0}+E_1\frac{\partial M_{12}}{\partial E_0}=0$$

and two other analogous equations. From these equations we find the coefficients A_2, B_2 and E_2. Further conditions are also linear equations in A_n, B_n and E_n. Thus the coefficients A_n, B_n and E_n are determined successively from systems of three linear equations with one and the same determinant equal to the above Jacobian. If this Jacobian goes to zero, then for the existence of periodic solutions, it is necessary that the Jacobian matrix and the expanded matrix resulting from the addition of a column with the free terms of the equations have one and the same rank. For the determination of the coefficients A_1, B_1 and E_1 one needs here an equation not lower than of second degree. Thus the vanishing of the Jacobian denotes either the absence of a periodic solution or a bifurcation of the generating solution.

If Equations (4.2) are satisfied identically, then the solvability of an infinite system of equations in A_n, B_n and E_n will be tied to the non-vanishing of the Jacobian

$$\frac{D(M_{12}, M_{22}, M_{32})}{D(A_0, B_0, E_0)} \qquad \text{etc.}$$

Once the coefficients A_n, B_n and E_n are known, one can find the correction of the period in the form of a power series in μ. We substitute the values of β_1, β_2 and β_3 into Formula (3.2) and collect terms of equal powers of μ. We obtain

$$\alpha = T_0 \sum_{n=1}^{\infty} h_n \mu^n \tag{4.5}$$

The first three coefficients h_1, h_2 and h_3 have the values

$$h_1 = \frac{1}{T_0} N_1(T_0), \qquad h_2 = \frac{1}{T_0}\left[N_2(T_0) + A_1 \frac{\partial N_1}{\partial A_0} + B_1 \frac{\partial N_1}{\partial B_0} + E_1 \frac{\partial N_1}{\partial E_0}\right] \tag{4.6}$$

$$h_3 = \frac{1}{T_0}\left[N_3(T_0) + A_2 \frac{\partial N_1}{\partial A_0} + B_2 \frac{\partial N_1}{\partial B_0} + E_2 \frac{\partial N_1}{\partial E_0} + \frac{1}{2} A_1^2 \frac{\partial^2 N_1}{\partial A_0^2} + \frac{1}{2} B_1^2 \frac{\partial^2 N_1}{\partial B_0^2} + \frac{1}{2} E_1^2 \frac{\partial^2 N_1}{\partial E_0^2} + \right.$$
$$\left. + A_1 B_1 \frac{\partial^2 N_1}{\partial A_0 \partial B_0} + A_1 E_1 \frac{\partial^2 N_1}{\partial A_0 \partial E_0} + B_1 E_1 \frac{\partial^2 N_1}{\partial B_0 \partial E_0} + A_1 \frac{\partial N_2}{\partial A_0} + B_1 \frac{\partial N_2}{\partial B_0} + E_1 \frac{\partial N_2}{\partial E_0}\right]$$

For the construction of a periodic solution of the system (1.1) with a period that is not dependent on the parameter μ we perform a change of the dependent variable by means of the formula

$$t = \tau(1 + h_1\mu + h_2\mu^2 + \ldots) \tag{4.7}$$

and we shall seek the solution in terms of the function τ. This solution will have a period equal to T_0.

By substituting t from Formula (4.7) into the functions $C_{in}(t)$, cos ωt and sin ωt and expanding them into series in μ we obtain

$$C_{in}(t) = C_{in}(\tau) + h_1\tau\dot{C}_{in}(\tau)\mu + \ldots$$

$$\cos \omega t = \cos \omega\tau - h_1\omega\tau \sin \omega\tau\, \mu - (h_2\omega\tau \sin \omega\tau + \tfrac{1}{2} h_1^2\omega^2\tau^2 \cos \omega\tau)\, \mu^2 + \ldots$$

$$\sin \omega t = \sin \omega\tau + h_1\omega\tau \cos \omega\tau\, \mu + (h_2\omega\tau \cos \omega\tau - \tfrac{1}{2} h_1^2\omega^2\tau^2 \sin \omega\tau)\mu^2 + \ldots$$

The functions $x_1(t)$ and $x_2(t)$ will be represented in the form of series of integral powers of the parameter μ

$$x_k(\tau) = x_{k0}(\tau) + \mu x_{k1}(\tau) + \mu^2 x_{k2}(\tau) + \ldots \qquad (k = 1, 2) \tag{4.8}$$

where

$$x_{1n}(\tau) = x_n^{(1)}(\tau) + x_n^{(2)}(\tau), \qquad x_{2n}(\tau) = p_1 x_n^{(1)}(\tau) + p_2 x_n^{(2)}(\tau) \tag{4.9}$$

The generating solution is determined by Formula (2.1). For the following two coefficients we obtain

$$\begin{aligned}
x_1^{(1)}(\tau) &= A_1 \cos \omega_1\tau + \frac{B_1}{\omega_1} \sin \omega_1\tau + C_1^{(1)}(\tau) - h_1\tau\,(A_0\omega_1 \sin \omega_1\tau - B_0 \cos \omega_1\tau) \\
x_1^{(2)}(\tau) &= E_0 \cos \omega_2\tau + C_1^{(2)}(\tau) - h_1\tau E_0\omega_2 \sin \omega_2\tau \\
x_2^{(1)}(\tau) &= A_2 \cos \omega_1\tau + \frac{B_2}{\omega_1} \sin \omega_1\tau + C_2^{(1)}(\tau) + A_1 \frac{\partial C_1^{(1)}}{\partial A_0} + B_1 \frac{\partial C_1^{(1)}}{\partial B_0} + E_1 \frac{\partial C_1^{(1)}}{\partial E_0} + \\
&\quad + h_1\tau\dot{C}_1^{(1)}(\tau) - \tau\,[\omega_1 (h_1A_1 + h_2A_0) \sin \omega_1\tau - (h_1B_1 + h_2B_0) \cos \omega_1\tau] - \\
&\quad - \frac{1}{2} h_1^2\tau^2\omega_1 (A_0\omega_1 \cos \omega_1\tau + B_0 \sin \omega_1\tau) \\
x_2^{(2)}(\tau) &= E_2 \cos \omega_2\tau + C_2^{(2)}(\tau) + A_1 \frac{\partial C_1^{(2)}}{\partial A_0} + B_1 \frac{\partial C_1^{(2)}}{\partial B_0} + E_1 \frac{\partial C_1^{(2)}}{\partial E_0} + h_1\tau C_1^{(2)}(\tau) - \\
&\quad - \tau\omega_2 (h_1E_1 + h_2E_0) \sin \omega_2\tau - \frac{1}{2} h_1^2\tau^2\omega_2^2 E_0 \cos \omega_2\tau
\end{aligned} \tag{4.10}$$

The problem of the determination of the radius of convergence of the series derived in this paper has not been studied.

5. Now we turn to the second case, where the frequencies ω_1 and ω_2 are different but non-commensurate. A periodic solution of the generating system can be achieved with one of these frequencies. We shall seek a periodic solution of the original nonlinear system (1.1), which, for instance, turns into the generating system for $\mu = 0$ with the frequency ω_1

$$x_{10}(t) = A_0 \cos \omega_1 t, \qquad x_{20}(t) = p_1 A_0 \cos \omega_1 t \tag{5.1}$$

This is a single-parameter family of solutions. The initial conditions for the system (1.1) will be in this case

$$x_1(0) = A_0 + \beta_1, \qquad \dot{x}_1(0) = 0, \qquad x_2(0) = p_1(A_0 + \beta_1), \qquad \dot{x}_2(0) = 0 \tag{5.2}$$

The given case is a special case of the previous one and the solution for it can be obtained from formulas derived for the first case by letting there

$$B_n = 0, \quad E_n = 0 \quad (n = 0, 1, 2, \ldots); \qquad C_{1n}(t) = C_n^{(1)}(t), \quad C_{2n}(t) = p_1 C_n^{(1)}(t)$$

Consequently, the solution of the original system (1.1) in the case of different but non-commensurate frequencies will be of the form

$$x_1(t) = x^{(1)}(t), \qquad x_2(t) = p_1 x^{(1)}(t) \tag{5.3}$$

The analysis of the given solution for systems with one degree of freedom [2] can be applied entirely to the present case.

6. Let us study the case of equal frequencies $\omega_1 = \omega_2 = \omega$. It is known that in this case the following relation holds between the coefficients of the equations of the generating system:

$$\frac{c_{11}}{a_{11}} = \frac{c_{12}}{a_{12}} = \frac{c_{22}}{a_{22}} = \omega^2$$

The original system (1.1) takes on the form

$$a_{11}(\ddot{x}_1 + \omega^2 x_1) + a_{12}(\ddot{x}_2 + \omega^2 x_2) = \mu F_1(x_1, x_2, \dot{x}_1, \dot{x}_2, \mu)$$
$$a_{21}(\ddot{x}_1 + \omega^2 x_1) + a_{22}(\ddot{x}_2 + \omega^2 x_2) = \mu F_2(x_1, x_2, \dot{x}_1, \dot{x}_2, \mu)$$

From this we obtain

$$\ddot{x}_1 + \omega^2 x_1 = \frac{\mu}{\Delta_0}(a_{22}F_1 - a_{12}F_2) = \mu F_1^*(x_1, x_2, \dot{x}_1, \dot{x}_2, \mu)$$
$$\ddot{x}_2 + \omega^2 x_2 = \frac{\mu}{\Delta_0}(a_{11}F_2 - a_{21}F_1) = \mu F_2^*(x_1, x_2, \dot{x}_1, \dot{x}_2, \mu) \tag{6.1}$$

Because of the autonomous quality of the system one can let $\dot{x}_2(0) = 0$. Then the solution of the generating system will have the form

$$x_{10}(t) = A_0 \cos \omega t + \frac{B_0}{\omega} \sin \omega t, \qquad x_{20}(t) = E_0 \cos \omega t \tag{6.2}$$

The initial conditions for the original system (1.1) will be

$$x_1(0) = A_0 + \beta_1, \qquad \dot{x}_1(0) = B_0 + \beta_2, \qquad x_2(0) = E_0 + \beta_3, \qquad \dot{x}_2(0) = 0 \tag{6.3}$$

In this case the expansions of the functions $x_1(t)$ and $x_2(t)$ in terms of the parameters β_1, β_2, β_3 and μ are of the form

$$x_1(t) = (A_0 + \beta_1)\cos\omega t + \frac{B_0 + \beta_2}{\omega}\sin\omega t + \sum_{n=1}^{\infty}\left[C_{1n}(t) + \frac{\partial C_{1n}}{\partial A_0}\beta_1 + \frac{\partial C_{1n}}{\partial B_0}\beta_2 + \frac{\partial C_{1n}}{\partial E_0}\beta_3 + \frac{1}{2}\frac{\partial^2 C_{1n}}{\partial A_0^2}\beta_1^2 + \ldots\right]\mu^n$$

$$x_2(t) = (E_0 + \beta_3)\cos\omega t + \sum_{n=1}^{\infty}\left[C_{2n}(t) + \frac{\partial C_{2n}}{\partial A_0}\beta_1 + \frac{\partial C_{2n}}{\partial B_0}\beta_2 + \frac{\partial C_{2n}}{\partial E_0}\beta_3 + \frac{1}{2}\frac{\partial^2 C_{2n}}{\partial A_0^2}\beta_1^2 + \ldots\right]\mu^n \tag{6.4}$$

The functions $C_{1n}(t)$ and $C_{2n}(t)$ can be found by means of the formula

$$C_{in}(t) = \frac{1}{\omega}\int_0^t H_{in}^*(t')\sin\omega(t - t')\,dt' \quad (i=1, 2) \tag{6.5}$$

The quantities $H_{in}^*(t)$ are determined by means of Formulas (2.11) where the functions F_i should be replaced by F_i^*.

In order to find the coefficients N_n and M_{jn} for a given case it is necessary to make the following substitutions in the above-obtained formulas for the case of different and commensurate frequencies:

$$P_1 = A_0\omega^2, \quad Q_1 = A_0\omega^4, \quad P_2 = E_0\omega^2, \quad Q_2 = E_0\omega^4, \quad p_1 = 0$$

The coefficients of the series which represent the solution $x_1(t)$ and $x_2(t)$ in the given case will equal

$$x_{1n}(\tau) = x_n^{(1)}(\tau), \qquad x_{2n}(\tau) = x_n^{(2)}(\tau) \tag{6.6}$$

Here, instead of the quantities $C_n^{(1)}(\tau)$ and $C_n^{(2)}(\tau)$ it is necessary to substitute the quantities $C_{1n}(\tau)$ and $C_{2n}(\tau)$, respectively, according to Formula (6.5), and ω_1 and ω_2 should be replaced by ω.

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